Anomalous scaling of a passive scalar advected by a turbulent velocity field with finite correlation time and uniaxial small-scale anisotropy

E. Jurčišinová[1,2] and M. Jurčišin[1,3]
[1]*Institute of Experimental Physics, Slovak Academy of Sciences, Watsonova 47, 040 01 Košice, Slovakia*
[2]*Laboratory of Information Technologies, Joint Institute for Nuclear Research, 141 980 Dubna, Moscow Region, Russia*
[3]*N.N. Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141 980 Dubna, Moscow Region, Russia*
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The influence of uniaxial small-scale anisotropy on the stability of the scaling regimes and on the anomalous scaling of the structure functions of a passive scalar advected by a Gaussian solenoidal velocity field with finite correlation time is investigated by the field theoretic renormalization group and operator product expansion within one-loop approximation. Possible scaling regimes are found and classified in the plane of exponents $\varepsilon-\eta$, where ε characterizes the energy spectrum of the velocity field in the inertial range $E\propto k^{1-2\varepsilon}$, and η is related to the correlation time of the velocity field at the wave number k which is scaled as $k^{-2+\eta}$. It is shown that the presence of anisotropy does not disturb the stability of the infrared fixed points of the renormalization group equations, which are directly related to the corresponding scaling regimes. The influence of anisotropy on the anomalous scaling of the structure functions of the passive scalar field is studied as a function of the fixed point value of the parameter u, which represents the ratio of turnover time of scalar field and velocity correlation time. It is shown that the corresponding one-loop anomalous dimensions, which are the same (universal) for all particular models with a concrete value of u in the isotropic case, are different (nonuniversal) in the case with the presence of small-scale anisotropy and they are continuous functions of the anisotropy parameters, as well as the parameter u. The dependence of the anomalous dimensions on the anisotropy parameters of two special limits of the general model, namely, the rapid-change model and the frozen velocity field model, are found when $u\to\infty$ and $u\to 0$, respectively.

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I. INTRODUCTION

One of the most interesting and still open question in fully developed turbulence is the theoretical explanation of possible deviations from the classical Kolmogorov-Obukhov theory which are suggested by both natural, as well as numerical experiments [1–4]. Such a behavior is encoded in the concepts intermittency and anomalous scaling. During the last two decades this problem was intensively studied within the scope of the models of passively advected scalar field (concentration of an admixture or temperature are examples) by a velocity field with prescribed Gaussian statistics. The reason is twofold. First, it is well known that the deviation from the classical Kolmogorov-Obukhov theory is even more strongly noticeable for passively advected scalar field than for the velocity field itself, see, e.g., Refs. [4–13], and second, the problem of passive advection of a scalar or vector field is considerably easier for theoretical investigation than the original problem of anomalous scaling in the framework of Navier-Stokes velocity field. On the other hand, these simplified models reproduce many of the anomalous features of genuine turbulent heat or mass transport observed in experiments, therefore they can be treated as the first step on the long way of the investigation of intermittency and anomalous scaling in fully developed turbulence.

The central role in the studies of passive advection was played by a simple model of a passive scalar quantity advected by a random Gaussian velocity field, white in time and self-similar in space, the so-called Kraichnan rapid-change model [14]. Namely, in the framework of the rapid-change model, for the first time, the anomalous scaling was established on the basis of a microscopic model [15] and corresponding anomalous exponents were calculated within controlled approximations [16,17] (see also survey paper [18], and references cited therein). In Refs. [16,17], the so-called zero-mode approach was applied and it was shown that nontrivial anomalous behavior is related to the zero modes (homogeneous solutions) of the closed system of exact differential equations satisfied by the equal-time correlation functions.

Another effective method for investigation of self-similar scaling behavior is the renormalization group (RG) technique [19–21], which can be also used in the theory of fully developed turbulence and related problems [22–24]. In Refs. [25,26], the field theoretic RG and operator-product expansion (OPE) was used in the systematic investigation of the Kraichnan's rapid-change model. It was shown that within the field theoretic RG approach the anomalous scaling is related to the existence in the model of the composite operators with negative critical dimensions in the OPE, which are usually termed as *dangerous* operators (see, e.g., Refs. [21,23,24] for details). The RG approach allows one to construct a systematic perturbation expansion for the anomalous exponents and to calculate them to the higher orders [25,26].

Afterwards, various descendants of the Kraichnan model, namely, models with inclusion of small-scale anisotropy [27], compressibility [28,29], the finite correlation time of velocity field [30–33], and helicity [34] were studied by the field theoretic RG approach. Moreover, advection of the passive vector field by the Gaussian self-similar velocity field (with and without large- and small-scale anisotropy, pressure, compressibility, and a finite correlation time) has been

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also investigated, all possible asymptotic scaling regimes and crossover among them have been classified, and anomalous scaling was analyzed [35] (see also survey paper [36]). A general conclusion of all these investigations is that the anomalous scaling, which is the most intriguing and important feature of the Kraichnan rapid-change model, remains valid for all generalized models.

The Kraichnan model works with white in time (δ correlated in time) and self-similar in space Gaussian statistics of the velocity field. In Ref. [30] the field theoretic RG technique and OPE method was applied in the analysis of a more general model of passively advected scalar field by a self-similar Gaussian velocity field with a finite correlation time first proposed in Ref. [7] (see also Refs. [37], the recent investigations in Refs. [38], as well as the corresponding part in Ref. [39]). This model contains the Kraichnan model as a special limit case (see the next section). Maybe the most interesting conclusion from the view of anomalous scaling analysis obtained in Ref. [30] is that within the one-loop approximation the anomalous behavior of all particular models of the general one (the Kraichnan model is an example) is the same, i.e., the corresponding critical dimensions associated with needed composite operators within the OPE are the same. This conclusion is held in the isotropic model, as well as in the model with large-scale anisotropy with incompressible (solenoidal) velocity field. This universality of the anomalous behavior is destroyed, e.g., by the assumption that velocity field is nonsolenoidal as was shown in Ref. [31], by two-loop approximation as was shown in Ref. [32] (there the anomalous exponents are nonuniversal as a result of their dependence on a dimensionless parameter u, the ratio of the velocity correlation time, and turnover time of scalar field), or by the assumption of the presence of small-scale anisotropy of the velocity field which will be demonstrated explicitly in the present work.

Thus, in what follows, we shall continue with the investigation of the model with a finite correlation time of velocity field [30] from the point of view of the influence of the uniaxial small-scale anisotropy on the anomalous scaling of the single-time structure functions of a passive scalar admixture. In contradistinction with Ref. [30], where the velocity was isotropic and the large-scale anisotropy was introduced by the imposed linear mean gradient, the uniaxial anisotropy in our model persists for all scales, leading to nonuniversality of the anomalous exponents through their dependence on the anisotropy parameters and ratio of characteristic time scales. It can be considered as an additional step to the construction of a more realistic model of anisotropic passive advection.

But first let us motivate such an investigation. The importance of this investigation is dictated by the question of the influence of anisotropy on inertial-range behavior of passively advected fields [17,27,30,31,40–45], as well as the velocity field itself [46–48] (see also the survey paper [49], and references cited therein, as well as recent astrophysical investigations, e.g, in Refs. [50,51]). On one hand, it was shown that for the even structure (or correlation) functions the exponents which describe the inertial range scaling exhibit universality and they are ordered hierarchically in respect to degree of anisotropy with leading contribution given by the exponent from the isotropic shell but, on the other hand, the survival of the anisotropy in the inertial range is demonstrated by the behavior of the odd structure functions, namely, the so-called skewness factor decreases down the scales slower than expected earlier in accordance with the classical Kolmogorov-Obukhov theory.

The aim of the present paper is twofold. First of all we shall find the dependence of the anomalous exponents on the anisotropy parameters of the model and on the parameter u, therefore we shall be able to answer the question whether the system with the finite time correlations of the velocity field with the presence of small-scale anisotropy is more anomalous than the Kraichnan model, i.e., whether the corresponding critical dimensions are less than those of the Kraichnan rapid-change model, which was investigated in Ref. [27]. The answer to this question can be treated as the first step on the way to the investigation of the model with velocity field driven by the stochastic Navier-Stokes equation, which is more complicated from a mathematical point of view. The second aim is to analyze whether the finite correlation time of the velocity field can lead to a more complicated structure of critical dimensions than was shown in Ref. [27] within the rapid-change model with small-scale anisotropy.

The main conclusions of the paper are the following: the anomalous scaling, which is universal for all models with a different value of the parameter u within the one-loop approximation in the isotropic model (or in the case with large-scale anisotropy), is strongly nonuniversal in the model with the presence of uniaxial small-scale anisotropy. The anomalous scaling exponents smoothly depend on the anisotropy parameters of the model, on the value of the parameter u, and on the space dimensionality d. Rather strong dependence on the anisotropy parameters is not universal even in the sense that there is different behavior for odd and even structure functions of the scalar field as functions of the parameter u and, at the same time, there is different behavior in different directions in the plane of anisotropy parameters for the structure function of a given order. On the other hand, the anomalous exponents, which are related to the eigenvalues of the corresponding matrix of critical exponents of important composite operators, are all real, i.e., no exotic behavior of the anomalous exponents is obtained.

However, it must be said that the Gaussian models with finite correlation time of velocity field, which are able to describe some features of genuine turbulence that are unreachable for the Kraichnan rapid-change model (e.g., helical effects cannot be investigated within the Kraichnan model [34]), have their own drawbacks. First of all it is their Galilean noninvariance [7]. As a consequence of this fact they do not take into account the self-advection of turbulent eddies. As a result of these so-called "sweeping effects" the different time correlations of the Eulerian velocity are not self-similar and depend strongly on the integral scale; see, e.g., Ref. [52] (see also Ref. [39]). Thus, as was discussed, e.g., in Ref. [30], the perturbative expansion in ε (ε characterizes the energy spectrum of the velocity field in the inertial range; see the next section) is potentially dangerous even in the Gaussian models with finite correlation time of the velocity field. It means that there exists a "critical" value ε_{crit} such that for $\varepsilon > \varepsilon_{crit}$ the critical dimension of the velocity field becomes negative, which leads to the appearance of new IR singularities in the corresponding diagrams. Physically, it means that a qualitative changeover in the behavior of the scalar field appears as was shown explicitly in Refs. [39,53,54] by direct computer simulations and nonperturbative methods. This is a serious problem of all models with a finite correlation time of velocity field and the value ε_{crit} can be considered as the upper bound on the range of validity of the model. In the Gaussian model under consideration the value of ε_{crit} is known exactly, namely, $\varepsilon_{crit}=1$ and it is stable under perturbative theory, i.e., it has no corrections in ε expansion (see the corresponding discussion in the following sections). On the other hand, as we shall discuss in the next section, the Kolmogorov "two-thirds law" for the spatial statistics of the velocity field corresponds to the value $\varepsilon=4/3$ of our model, which is out of range of the validity of the model. Nevertheless, the model is interesting enough to be studied as a further nontrivial step to the investigation of real turbulent advection processes and can be used for demonstration of the fact that the anomalous exponents can depend on more details of the velocity field statistics than it is possible to investigate in the framework of the rapid-change model.

In the end, let us describe briefly the solution of the problem in the framework of the field theoretic approach [21,23,24]. It can be divided into two main stages. On the first stage the multiplicative renormalizability of the corresponding field theoretic model is demonstrated and the differential RG equations for its correlation functions are obtained. The asymptotic behavior of the latter on their ultraviolet argument (r/ℓ) for $r \gg \ell$ and any fixed (r/L) is given by infrared stable fixed points of those equations. Here ℓ and L are inner (ultraviolet) and outer (infrared) scales (lengths). It involves some "scaling functions" of the infrared argument (r/L), whose form is not determined by the RG equations. On the second stage, their behavior at $r \ll L$ is found from the OPE within the framework of the general solution of the RG equations. There, the crucial role is played by the critical dimensions of various composite operators, which give rise to an infinite family of independent aforementioned scaling exponents (and hence to multiscaling).

The paper is organized as follows. In Sec. II, the field theoretic formulation of the model is given. In Sec. III, we perform the ultraviolet (UV) renormalization of the model. In Sec. IV, we discuss the stability of possible scaling regimes of the model. In Sec. V, the dependence of anomalous exponents on parameters of the model is found. Obtained results are reviewed and discussed in Sec. VI.

II. FIELD THEORETIC DESCRIPTION OF THE MODEL

The advection of a passive scalar field $\theta(x) \equiv \theta(t,\mathbf{x})$ in an incompressible turbulent environment is described by the stochastic equation

$$\partial_t \theta + v_i \partial_i \theta = \nu_0 \Delta \theta + f, \tag{1}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, ν_0 is the molecular diffusivity coefficient (in what follows, a subscript 0 denotes bare parameters of unrenormalized theory), $\Delta \equiv \partial^2$ is the Laplace operator, $v_i \equiv v_i(x)$ is the ith component of the divergence-free (owing to the incompressibility) velocity field $\mathbf{v}(x)$, and $f \equiv f(x)$ is an artificial Gaussian random noise with zero mean and correlation function

$$D^\theta(x;x') = \langle f(x) f(x') \rangle = \delta(t-t') C(\mathbf{r}/L), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}', \tag{2}$$

where the angular brackets $\langle \dots \rangle$ hereafter denote the average over the corresponding statistical ensemble and L is an integral scale related to the stirring. The random noise is introduced to maintain the steady state of the system but the detailed form of the function $C(\mathbf{r}/L)$ in Eq. (2) will be inessential in our consideration. The only condition which must be satisfied by the function $C(\mathbf{r}/L)$ is that it must be finite and must decrease rapidly for $r \gg L$. In the problems related to the genuine turbulence the velocity field $\mathbf{v}(x)$ satisfies the Navier-Stokes equation but, in what follows, we shall work with a simplified model where we suppose that the statistics of the velocity field is given in the form of a Gaussian distribution with zero mean and pair correlation function [30,31]

$$D^v_{ij}(x;x') = \langle v_i(x) v_j(x') \rangle = \int \frac{d\omega d^d\mathbf{k}}{(2\pi)^{d+1}} P_{ij}(\mathbf{k}) \times D^v(\omega,k) e^{-i[\omega(t-t')-\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]}, \tag{3}$$

with

$$D^v(\omega,k) = \frac{D_0 k^{4-d-2\varepsilon-\eta}}{(i\omega + u_0\nu_0 k^{2-\eta})(-i\omega + u_0 \nu_0 k^{2-\eta})}, \tag{4}$$

where $k=|\mathbf{k}|$ and a transverse projector $P_{ij}(\mathbf{k})$ reflects the vectorial nature of the solenoidal velocity field. In the isotropic case it has the form of the simple transverse projector

$$P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}. \tag{5}$$

In the anisotropic case the transverse projector becomes more complicated as it will be specified below (see also Ref. [27]). In Eq. (4), $D_0 \equiv g_0 \nu_0^3$ is a positive amplitude factor and the introduced parameter g_0 plays the role of the coupling constant of the model. In addition, g_0 is a formal small parameter of the ordinary perturbation theory. On the other hand, the parameter u_0, introduced in the denominator of Eq. (4), gives the ratio of turnover time of scalar field and velocity correlation time (see, e.g., Ref. [30] for details). The positive exponents ε and η $[\varepsilon = O(\eta)]$ are small RG expansion parameters. Thus, we have a kind of double expansion model in the $\varepsilon-\eta$ plane around the origin $\varepsilon=\eta=0$. The coupling constant g_0 and the exponent ε control the behavior of the equal-time pair correlation function of velocity field (mean square velocity) or, equivalently, energy spectrum. On the other hand, the parameter u_0 and the second exponent η are related to the frequency $\omega \simeq u_0 \nu_0 k^{2-\eta}$, which characterizes the mode k [30,53–56]. Thus, in our notation, the value $\varepsilon=4/3$ corresponds to the celebrated Kolmogorov "two-thirds law" for the spatial statistics of the velocity field or, equivalently, "five-thirds law" for the energy spectrum, and $\eta = 4/3$ corresponds to the Kolmogorov frequency. Simple dimensional analysis shows that g_0 and u_0, which we commonly term as charges, are related to the characteristic ultraviolet (UV) momentum scale Λ (or inner length $l \sim \Lambda^{-1}$) by the following relations:

$$g_0 \simeq \Lambda^{2\varepsilon+\eta}, \quad u_0 \simeq \Lambda^{\eta}. \tag{6}$$

As was discussed in the Introduction, in the present paper, we shall take the velocity statistics to be anisotropic at all scales. For that purpose, we replace the ordinary transverse projector $P_{ij}(\mathbf{k})$ in Eq. (3) with the general uniaxially anisotropic transverse tensor structure (see, e.g., Ref. [27]) as follows:

$$T_{ij}(\mathbf{k}) = a(\psi)P_{ij}(\mathbf{k}) + b(\psi)P_{is}(\mathbf{k})n_s n_t P_{tj}(\mathbf{k}), \tag{7}$$

where n_i is the ith component of the unit vector $\mathbf{n}$ ($\mathbf{n}^2=1$), which determines the distinguished direction of uniaxial anisotropy and ψ is the angle between the vectors $\mathbf{k}$ and $\mathbf{n}$, so that $\mathbf{n}\cdot\mathbf{k}=k\cos\psi$. It is well known that functions $a(\psi)$ and $b(\psi)$ can be decomposed into the d-dimensional generalization of the Legendre polynomials, which are known as the Gegenbauer polynomials [57], namely,

$$a(\psi) = \sum_{l=0}^{\infty} a_l P_{2l}(\cos\psi), \quad b(\psi) = \sum_{l=0}^{\infty} b_l P_{2l}(\cos\psi) \tag{8}$$

(as was shown in Ref. [27] the odd polynomials do not affect the scaling behavior). The necessary condition to have the positively defined velocity correlator (3) leads to the following inequalities for these functions [27]:

$$a(\psi) > 0, \quad a(\psi) + b(\psi)\sin^2\psi > 0. \tag{9}$$

But in practical calculations it is impossible to work with the general tensor structure as is defined in Eq. (7). The reason is, at least, because it contains an infinite number of parameters a_i and b_j in the corresponding decomposition (8). Therefore, in what follows, we shall work with the simplest special case of the general uniaxial anisotropic transverse projector, namely,

$$T_{ij}(\mathbf{k}) = \left(1 + \alpha_1 \frac{(\mathbf{n}\cdot\mathbf{k})^2}{k^2}\right)P_{ij}(\mathbf{k}) + \alpha_2 P_{is}(\mathbf{k})n_s n_t P_{tj}(\mathbf{k}), \tag{10}$$

which is sufficient for investigation of principal properties of the uniaxial anisotropy (see the corresponding discussion in Ref. [27]). In this case, the inequalities (9) reduce into the requirements $\alpha_1 > -1, \alpha_2 > -1$.

The transverse tensor defined in Eq. (10) corresponds to the uniaxial anisotropic nonhelical situation. It can be also generalized to the anisotropic helical case, which allows one to investigate effects related to helicity. But, as was shown in Refs. [34], to be able to study the helicity effects it is necessary to work at least in two-loop approximation in the framework of the field theoretic RG approach. Because, in what follows, we shall analyze the model in one-loop approximation (see Sec. III) the corresponding helical tensor structures are irrelevant. Thus, we do not introduce them in Eq. (10).

Let us briefly discuss two special limits of the considered model (3),(4) (see also Ref. [30]). The first of them is the so-called rapid-change model limit when $u_0 \to \infty$ and $g_0' \equiv g_0/u_0^2 =$ const,

$$D^v(\omega,k) \to g_0' \nu_0 k^{-d-2\varepsilon+\eta}, \tag{11}$$

and the second is the so-called quenched (time-independent or frozen) velocity field limit, which is defined by $u_0 \to 0$ and $g_0'' \equiv g_0/u_0 =$ const,

$$D^v(\omega,k) \to g_0'' \nu_0^2 \pi \delta(\omega) k^{-d+2-2\varepsilon}, \tag{12}$$

which is similar to the well-known models of random walks in a random environment with long-range correlations; see, e.g., Refs. [58,59].

Using the well-known Martin-Siggia-Rose mechanism [60] (see also, e.g., Refs. [20,21]) the stochastic problems (1)–(4) in the presence of small-scale uniaxial anisotropy defined by the transverse projector (10) in the velocity field correlator (3) can be treated as a field theory with action functional

$$\begin{aligned} S(\theta,\theta',\mathbf{v}) = &-\frac{1}{2}\int dt_1 d^d\mathbf{x_1} dt_2 d^d\mathbf{x_2} v_i(t_1,\mathbf{x_1}) \\ &\times [D^v_{ij}(t_1,\mathbf{x_1};t_2,\mathbf{x_2})]^{-1} v_j(t_2,\mathbf{x_2}) \\ &+\frac{1}{2}\int dt_1 d^d\mathbf{x_1} dt_2 d^d\mathbf{x_2} \\ &\times \theta'(t_1,\mathbf{x_1}) D^\theta(t_1,\mathbf{x_1};t_2,\mathbf{x_2}) \theta'(t_2,\mathbf{x_2}) \\ &+\int dt d^d\mathbf{x}\, \theta'[-\partial_t - v_i\partial_i + \nu_0\Delta + \chi_0\nu_0(\mathbf{n}\cdot\partial)^2]\theta, \end{aligned} \tag{13}$$

where θ' is an auxiliary scalar field, and D^θ and D^v are correlators (2) and (3), respectively, with the substitution of P_{ij} by T_{ij} as defined in Eq. (10). The term $\chi_0\nu_0(\mathbf{n}\cdot\partial)^2$ with a new unrenormalized parameter χ_0, which is not present in the original stochastic equation (1) is related to uniaxial anisotropy and it must be introduced into the action to make the model multiplicatively renormalizable (see, e.g., Ref. [27] for details). The stability of the system implies the positivity of the total viscous contribution $\nu_0 k^2 + \chi_0\nu_0(\mathbf{nk})^2$, which leads to the condition $\chi_0 > -1$. In action (13) all required summations over the vector indices are understood. The second and the third integral in Eq. (13) represent the De Dominicis–Janssen-type action for the stochastic problem (1), (2) at fixed $\mathbf{v}$, and the first integral represents the Gaussian averaging over $\mathbf{v}$.

Model (13) corresponds to a standard Feynman diagrammatic technique with the bare propagators $\langle\theta\theta'\rangle_0$ and $\langle v_i v_j\rangle_0$ (in the frequency-momentum representation),

$$\langle\theta(\omega,\mathbf{k})\theta'(\omega,\mathbf{k})\rangle_0 = \frac{1}{-i\omega + \nu_0 k^2 + \chi_0\nu_0(\mathbf{nk})^2}, \tag{14}$$

$$\langle v_i(\omega,\mathbf{k}) v_j(\omega,\mathbf{k})\rangle_0 = T_{ij}(\mathbf{k}) D^v(\omega,k), \tag{15}$$

where $D^v(\omega,k)$ is given directly by Eq. (4) and tensor $T_{ij}(\mathbf{k})$ is given in Eq. (10). In the Feynman diagrams these propa-

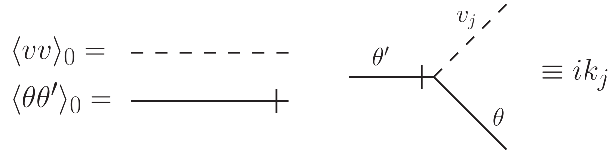

FIG. 1. (Left) Graphical representation of needed propagators of the model. (Right) The triple (interaction) vertex of the model. Momentum $\mathbf{k}$ is flowing into the vertex via the auxiliary field θ'.

gators are represented by the lines which are shown in Fig. 1 (the end with a slash in the propagator $\langle\theta\theta'\rangle_0$ corresponds to the field θ', and the end without a slash corresponds to the field θ). The triple vertex (or interaction vertex) $-\theta' v_j \partial_j \theta = \theta' v_j V_j \theta$, where $V_j = ik_j$ (in the momentum-frequency representation), is present in Fig. 1, where momentum $\mathbf{k}$ is flowing into the vertex via the auxiliary field θ'.

The formulation of the problem through the action functional (13) replaces the statistical averages of random quantities in the stochastic problem defined by Eqs. (1)–(3) with equivalent functional averages with weight $\exp S(\Phi)$, where $\Phi = \{\theta, \theta', \mathbf{v}\}$. The generating functionals of the total Green functions $G(A)$ and connected Green functions $W(A)$ are then defined by the functional integral

$$G(A) = e^{W(A)} = \int \mathcal{D}\Phi e^{S(\Phi)+A\Phi}, \tag{16}$$

where $A(x) = \{A^{\theta}, A^{\theta'}, \mathbf{A}^{\mathbf{v}}\}$ represents a set of arbitrary sources for the set of fields Φ, $\mathcal{D}\Phi \equiv \mathcal{D}\theta \mathcal{D}\theta' \mathcal{D}\mathbf{v}$ denotes the measure of functional integration, and the linear form $A\Phi$ is defined as

$$A\Phi = \int dx [A^{\theta}(x)\theta(x) + A^{\theta'}(x)\theta'(x) + A_i^{v}(x) v_i(x)]. \tag{17}$$

III. RENORMALIZATION GROUP ANALYSIS

Using the standard analysis of canonical dimensions leads to the information about possible UV divergences in the model (see, e.g., Refs. [20,21]). The dynamical model (13) belongs to the class of the so-called two-scale models [21,23,24], i.e., to the class of models for which the canonical dimension of some quantity F is given by two numbers, namely, the momentum dimension d_F^k and the frequency dimension d_F^{ω}. To find the dimensions of all quantities it is convenient to use the standard normalization conditions $d_k^k = -d_x^k = 1$, $d_{\omega}^{\omega} = -d_t^{\omega} = 1$, $d_k^{\omega} = d_x^{\omega} = d_{\omega}^k = d_t^k = 0$, and the requirement that each term of the action functional must be dimensionless separately with respect to the momentum and frequency dimensions. The total canonical dimension d_F is then defined as $d_F = d_F^k + 2d_F^{\omega}$ (it is related to the fact that $\partial_t \propto \nu_0 \partial^2$ in the free action (13) with a choice of zero canonical dimension for ν_0). In the framework of the theory of renormalization the total canonical dimension in dynamical models plays the same role as the momentum dimension does in static models.

The canonical dimensions of our model are present in Table I, where also the canonical dimensions of the renormalized parameters are shown. The model (13) is logarithmic at $\varepsilon = \eta = 0$ (the coupling constants g_0 and u_0 are dimensionless); therefore, in the framework of the minimal substraction (MS) scheme [20], which is always used in what follows, possible UV divergences in the correlation functions have the form of poles in ε, η, and their linear combinations. It is well known that the superficial divergences can be present only in the one-irreducible Green functions for which the corresponding total canonical dimensions are a non-negative integer. A detailed analysis of the possible divergences was done, e.g., in Ref. [27], therefore we shall not repeat it here. This analysis shows that a superficially divergent function of our model is only function $\langle\theta'\theta\rangle_{one\text{-}ir}$ and the action (13) has all the necessary tensor structures to remove divergences multiplicatively (for details see Ref. [27]). All divergences can be removed by the counterterms of the forms $\theta'\Delta\theta$ and $\theta'(\mathbf{n}\cdot\partial)^2\theta$, which can be explicitly expressed in the multiplicative renormalization of the parameters g_0, u_0, ν_0, and χ_0 in the form

$$\nu_0 = \nu Z_{\nu}, \quad g_0 = g\mu^{2\varepsilon+\eta} Z_g, \quad u_0 = u\mu^{\eta} Z_u, \quad \chi_0 = \chi Z_{\chi}. \tag{18}$$

Here the dimensionless parameters g, u, ν, and χ are the renormalized counterparts of the corresponding bare ones, μ is the renormalization mass (a scale-setting parameter), an artifact of the dimensional regularization. Quantities $Z_i = Z_i(g, u, \chi; d; \varepsilon, \eta)$ are the so-called renormalization constants and, in general, they contain poles in linear combinations of ε and η.

The renormalized action functional has the following form:

$$\begin{aligned} S_R(\theta, \theta', \mathbf{v}) = &-\frac{1}{2}\int dt_1 d^d\mathbf{x_1} dt_2 d^d\mathbf{x_2} v_i(t_1, \mathbf{x_1}) \\ &\times [D_{ij}^{v}(t_1, \mathbf{x_1}; t_2, \mathbf{x_2})]^{-1} v_j(t_2, \mathbf{x_2}) \\ &+ \frac{1}{2}\int dt_1 d^d\mathbf{x_1} dt_2 d^d\mathbf{x_2} \\ &\times \theta'(t_1, \mathbf{x_1}) D^{\theta}(t_1, \mathbf{x_1}; t_2, \mathbf{x_2}) \theta'(t_2, \mathbf{x_2}) \end{aligned}$$

TABLE I. Canonical dimensions of the fields and parameters of the model under consideration.

F	$\mathbf{v}$	θ	θ'	m, Λ, μ	ν_0, ν	g_0	u_0	g, u, χ_0, χ
d_F^k	-1	0	d	1	-2	$2\varepsilon+\eta$	η	0
d_F^{ω}	1	$-1/2$	$1/2$	0	1	0	0	0
d_F	1	-1	$d+1$	1	0	$2\varepsilon+\eta$	η	0

$$+\int dt d^d\mathbf{x}\,\theta'[-\partial_t - v_i\partial_i + \nu Z_1\Delta + \chi\nu Z_2(\mathbf{n}\cdot\partial)^2]\theta. \qquad (19)$$

By comparison of the renormalized action (19) with definitions of the renormalization constants Z_i, $i=g,u,\nu,\chi$, which are given in Eqs. (18), we come to the relations among them,

$$Z_\nu = Z_1, \quad Z_\chi = Z_2 Z_1^{-1}, \quad Z_g = Z_1^{-3}, \quad Z_u = Z_1^{-1}. \qquad (20)$$

The issue of interest is, in particular, the behavior of response functions, e.g., $\langle\theta(x)\theta'(x')\rangle$, correlation functions $\langle\theta(x_1)\theta(x_2)\cdots\theta(x_n)\rangle$, and the equal-time structure functions

$$S_N(r) \equiv \langle[\theta(t,\mathbf{x}) - \theta(t,\mathbf{x}')]^N\rangle, \quad r=|\mathbf{x}-\mathbf{x}'| \qquad (21)$$

in the inertial range specified by the inequalities $l\sim 1/\Lambda \ll r \ll L=1/m$ (l is an internal length). In the field theoretic formulation of our stochastic problem the angular brackets $\langle\ldots\rangle$ mean functional average over fields $\theta,\theta',\mathbf{v}$ with weight $\exp(S_R)$. Independence of the original unrenormalized model of the scale-setting parameter μ of the renormalized model yields the RG differential equations for the renormalized correlation functions of the fields, e.g.,

$$\left[\mathcal{D}_\mu + \sum_{i=g,\chi,u}\beta_i\partial_i - \gamma_\nu\mathcal{D}_\nu\right]\langle\theta(\mathbf{x},t)\theta(\mathbf{x}',t')\rangle_R = 0. \qquad (22)$$

Here $\mathcal{D}_x \equiv x\partial_x$ stands for any variable x and the RG functions (the β and γ functions) are given by the well known definitions [20,21]. In our case, using the relations (20) for the renormalization constants, they acquire the following form:

$$\gamma_i \equiv \mathcal{D}_\mu \ln Z_i \qquad (23)$$

for any renormalization constant Z_i, and

$$\beta_g \equiv \mathcal{D}_\mu g = g(-2\varepsilon - \eta + 3\gamma_1), \qquad (24)$$

$$\beta_u \equiv \mathcal{D}_\mu u = u(-\eta + \gamma_1), \qquad (25)$$

$$\beta_\chi \equiv \mathcal{D}_\mu \chi = \chi(\gamma_1 - \gamma_2). \qquad (26)$$

The renormalization constants Z_1 and Z_2 are determined by the requirement that the one-particle irreducible Green function $\langle\theta'\theta\rangle_{one\text{-}ir}$ must be UV finite when it is written in the renormalized variables. In our case this means that it has no singularities in the limit $\varepsilon,\eta\to 0$. The one-particle irreducible Green function $\langle\theta'\theta\rangle_{one\text{-}ir}$ is related to the self-energy operator $\Sigma_{\theta'\theta}$, which is expressed via Feynman graphs, by the Dyson equation. In frequency-momentum representation it has the following form:

$$\langle\theta'\theta\rangle_{one\text{-}ir} = -i\omega + \nu_0 p^2 + \nu_0\chi_0(\mathbf{n}\cdot\mathbf{p})^2 - \Sigma_{\theta'\theta}(\omega,p). \qquad (27)$$

Thus Z_1 and Z_2 are found from the requirement that the UV divergences are canceled in Eq. (27) after the substitution $\nu_0=\nu Z_\nu$, $\chi_0=\chi Z_\chi$. This determines Z_1 and Z_2 up to an UV finite contribution, which is fixed by the choice of the renormalization scheme. In the MS scheme all the renormalization constants have the form: $1+$*poles in* ε, η *and their linear combinations*. In one-loop approximation the self-energy operator $\Sigma_{\theta'\theta}$ is defined by the Feynman diagram, which is shown in Fig. 2.

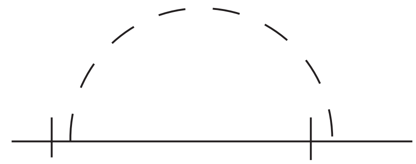

FIG. 2. The one-loop diagram that contributes to the self-energy operator $\Sigma_{\theta'\theta}$.

It can be shown that in one-loop calculations it is enough to work with $\eta=0$ (see, e.g., Refs. [30–32] for details). This possibility essentially simplifies the evaluations of all quantities. Then the divergent part of the diagram given in Fig. 2 has only poles in ε. Its explicit analytical form is given as follows (in renormalized parameters and within one-loop approximation):

$$\Sigma_{\theta'\theta}(p) = -\frac{S_d}{(2\pi)^d}\frac{g\nu}{2u(1+u)}\frac{1}{d(d+2)}\frac{1}{\varepsilon}\times[p^2 A + (\mathbf{n}\cdot\mathbf{p})^2 B], \qquad (28)$$

with

$$A = (1+\alpha_1)d(d+2)\,{}_2F_1\left(1;\frac{1}{2};\frac{d}{2};\frac{-\chi}{1+u}\right) + (\alpha_2 - \alpha_1 d - 1)(d+2)\,{}_2F_1\left(1;\frac{1}{2};1+\frac{d}{2};\frac{-\chi}{1+u}\right) + (\alpha_1-\alpha_2)(d+1)\,{}_2F_1\left(1;\frac{1}{2};2+\frac{d}{2};\frac{-\chi}{1+u}\right), \qquad (29)$$

$$B = -(1+\alpha_1)d(d+2)\,{}_2F_1\left(1;\frac{1}{2};\frac{d}{2};\frac{-\chi}{1+u}\right) - [\alpha_1(1-2d)+\alpha_2-d](d+2)\,{}_2F_1\left(1;\frac{1}{2};1+\frac{d}{2};\frac{-\chi}{1+u}\right) - (\alpha_1-\alpha_2)d(d+1)\,{}_2F_1\left(1;\frac{1}{2};2+\frac{d}{2};\frac{-\chi}{1+u}\right), \qquad (30)$$

where $S_d=2\pi^{d/2}/\Gamma(d/2)$ denotes the surface of the d-dimensional unit sphere and ${}_2F_1(a,b,c,z)=1+\frac{ab}{c\cdot 1}z+\frac{a(a+1)b(b+1)}{c(c+1)\cdot 1\cdot 2}z^2+\cdots$ represents the corresponding hypergeometric function.

In the end, the renormalization constants Z_1 and Z_2 are given as follows:

$$Z_1 = 1 - \frac{\bar{g}}{2u(1+u)}\frac{1}{d(d+2)}\frac{A}{\varepsilon}, \qquad (31)$$

$$Z_2 = 1 - \frac{\bar{g}}{2u(1+u)}\frac{1}{d(d+2)\chi}\frac{B}{\varepsilon}, \qquad (32)$$

where we have introduced new notation $\bar{g}=gS_d/(2\pi)^d$.

Now using the definition of the anomalous dimensions γ_1 and γ_2 in Eq. (23) one comes to the following expressions:

$$\gamma_1 = \frac{\bar{g}}{2u(1+u)d(d+2)}A, \tag{33}$$

$$\gamma_2 = \frac{\bar{g}}{2u(1+u)d(d+2)\chi}B. \tag{34}$$

In the next section we shall use these results for the investigation of possible scaling regimes of the model.

IV. FIXED POINTS AND SCALING REGIMES

Possible scaling regimes of a renormalized model are directly given by the infrared (IR) stable fixed points of the corresponding system of the RG equations [20,21]. The fixed point of the RG equations is defined by β functions, namely, by requirement of their vanishing. In our model the coordinates g_*, u_*, χ_* of all possible fixed points are found from the system of three equations

$$\beta_g(g_*,u_*,\chi_*) = \beta_u(g_*,u_*,\chi_*) = \beta_\chi(g_*,u_*,\chi_*) = 0. \tag{35}$$

The β functions β_g, β_u, and β_χ are defined in Eqs. (24)–(26). To investigate the IR stability of a fixed point it is enough to analyze the eigenvalues of the matrix Ω of the first derivatives as follows:

$$\Omega_{ij} = \begin{pmatrix} \partial\beta_g/\partial g & \partial\beta_g/\partial u & \partial\beta_g/\partial\chi \\ \partial\beta_u/\partial g & \partial\beta_u/\partial u & \partial\beta_u/\partial\chi \\ \partial\beta_\chi/\partial g & \partial\beta_\chi/\partial u & \partial\beta_\chi/\partial\chi \end{pmatrix}. \tag{36}$$

The IR asymptotic behavior is governed by the IR stable fixed points, i.e., those for which real parts of all eigenvalues are non-negative.

First of all, we shall study the rapid-change model limit: $u\to\infty$. In this regime, it is convenient to make a transformation to new variables, namely, $w \equiv 1/u$, and $g' \equiv g/u^2$ [30], with the corresponding changes in the β functions as follows:

$$\beta_{g'} = g'(-2\varepsilon + \eta + \gamma_1), \tag{37}$$

$$\beta_w = w(\eta - \gamma_1), \tag{38}$$

while β_χ is unchanged, i.e., it is given by Eq. (26).

In the rapid-change model limit $w\to 0$ $(u\to\infty)$ we are coming to the result of Refs. [27] with the anomalous dimensions γ_1 and γ_2 of the form

$$\gamma_1 = \frac{\bar{g}'}{2d(d+2)}[(d-1)(d+2) + \alpha_1(d+1) + \alpha_2], \tag{39}$$

$$\gamma_2 = \frac{\bar{g}'}{2d(d+2)\chi}[-2\alpha_1 + (d^2-2)\alpha_2], \tag{40}$$

where again $\bar{g}' = g' S_d/(2\pi)^d$. For completeness we shall briefly discuss this special case. In this limit we have two fixed points denoted as FPI and FPII. The first fixed point is trivial,

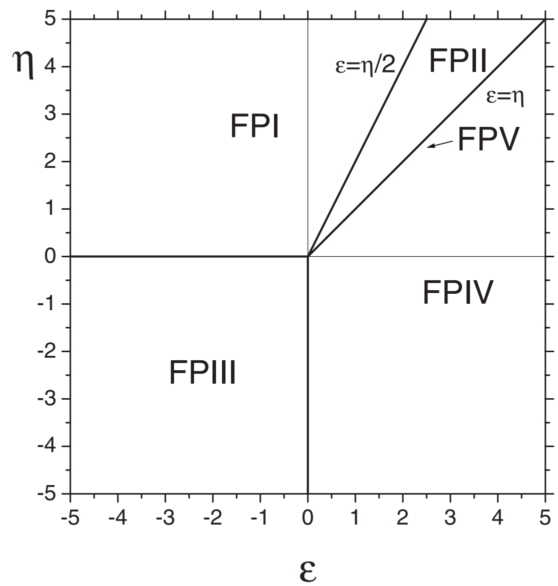

FIG. 3. The "phase" diagram of the fixed points of the model (see the text for details).

$$\text{FPI:} \quad w_* = g'_* = 0, \tag{41}$$

with arbitrary χ_* and $\gamma_1^*=0$, $\gamma_2^*=0$. The corresponding "stability matrix" is triangular with diagonal elements (eigenvalues) as follows:

$$\lambda_1 = -2\varepsilon + \eta, \quad \lambda_2 = \eta, \quad \lambda_3 = 0. \tag{42}$$

The region of the IR stability is shown in Fig. 3. The second point is defined as

$$\text{FPII:} \quad w_* = 0, \tag{43}$$

$$\bar{g}'_* = \frac{2d(d+2)(2\varepsilon-\eta)}{(d+2)(d-1) + \alpha_1(d+1) + \alpha_2}, \tag{44}$$

$$\chi_* = \frac{-2\alpha_1 + \alpha_2(d^2-2)}{(d+2)(d-1) + \alpha_1(d+1) + \alpha_2}, \tag{45}$$

with $\gamma_1^* = \gamma_2^* = 2\varepsilon - \eta$. The triangular matrix Ω has the following eigenvalues (diagonal elements):

$$\lambda_1 = 2\varepsilon - \eta, \quad \lambda_2 = 2\varepsilon - \eta, \quad \lambda_3 = -2\varepsilon + 2\eta. \tag{46}$$

The region of the IR stability of this fixed point is shown in Fig. 3.

Now let us analyze the "frozen regime" with frozen velocity field. It is mathematically obtained from the model under consideration in the limit $u\to 0$. To study this transition it is appropriate to change the variable g to the new variable $g'' \equiv g/u$ [30]. Then the β_g function is transformed to the following one:

$$\beta_{g''} = g''(-2\varepsilon + 2\gamma_1), \tag{47}$$

while β_u and β_χ functions are not changed, i.e., they are the same as the initial ones given by Eqs. (25) and (26).

In the limit $u\to 0$ the anomalous dimensions γ_1 and γ_2 acquire the following form:

$$\gamma_1 = \frac{\bar{g}''}{2d(d+2)}A'', \tag{48}$$

$$\gamma_2 = \frac{\bar{g}''}{2d(d+2)\chi}B'', \tag{49}$$

where $\bar{g}''=g''S_d/(2\pi)^d$ and A'' and B'' are given by Eqs. (29) and (30) with $u=0$.

The system of β functions (25), (26), and (47), exhibits two fixed points, denoted as FPIII and FPIV, related to the corresponding two scaling regimes. One of them is again trivial, namely,

$$\text{FPIII:} \quad u_* = g''_* = 0, \tag{50}$$

with arbitrary χ_* and $\gamma_1^*=\gamma_2^*=0$. The eigenvalues of the corresponding matrix Ω are

$$\lambda_1 = -2\varepsilon, \quad \lambda_2 = -\eta, \quad \lambda_3 = 0. \tag{51}$$

Thus this regime is IR stable only if both parameters ε and η are negative simultaneously as can be seen in Fig. 3. The second, nontrivial, point is

$$\text{FPIV:} \quad u_* = 0, \tag{52}$$

$$\bar{g}''_* = \frac{2d(d+2)\varepsilon}{A''_*}, \tag{53}$$

where A''_* is A'' taken at the fixed point, i.e., χ is replaced by χ_*, which is given only implicitly by the equation

$$\chi_* A''_* - B''_* = 0, \tag{54}$$

where B''_* is B'' taken at the fixed point.

Straightforward analysis shows that to have $\bar{g}''_*>0$ together with $\chi_*>-1$ one must suppose $\varepsilon>0$. It is the only condition related to the coordinates of the fixed point. The IR stability of the fixed point is again given by the Ω matrix, namely, by the positive values of real parts of its eigenvalues. It is triangular in this case, thus its eigenvalues are given directly by the diagonal elements. The eigenvalues are

$$\lambda_1 = 2\varepsilon, \tag{55}$$

$$\lambda_2 = \varepsilon - \eta, \tag{56}$$

$$\lambda_3 = \chi_* \left(\frac{\partial \gamma_1}{\partial \chi} - \frac{\partial \gamma_2}{\partial \chi} \right)_*. \tag{57}$$

Here λ_3 has a rather complicated explicit form but it can be numerically shown that λ_3 is always positive for $\alpha_{1,2}>-1$, $\varepsilon>0$, and $d>0$. The region of stability of this fixed point is shown in Fig. 3.

Now let us turn to the most interesting scaling regime with a finite value of the fixed point for the variable u. By short analysis one immediately concludes that the system of equations

$$\beta_g = g(-2\varepsilon - \eta + 3\gamma_1) = 0, \tag{58}$$

$$\beta_u = u(-\eta + \gamma_1) = 0, \tag{59}$$

$$\beta_\chi = \chi(\gamma_1 - \gamma_2) = 0, \tag{60}$$

can be fulfilled simultaneously for finite values of g and u only when the parameter ε is equal to η: $\varepsilon=\eta$. In this case the function β_g is proportional to function β_u. As a result we have not one fixed point but a set of fixed points g_*, χ_* that depend on arbitrary parameter $u_*>0$. The value of the fixed point for the variable g in one-loop approximation is given as follows (we denote it as FPV):

$$\text{FPV:} \quad g_* = \frac{2u(1+u_*)d(d+2)\varepsilon}{A_*}, \tag{61}$$

where A_* is A from Eq. (29) with u and χ is replaced by u_* and χ_*, respectively. On the other hand, χ_* is again known only implicitly and it can be obtained from Eq. (60), which is equivalent to the condition

$$\gamma_1^* = \gamma_2^*, \tag{62}$$

where γ_1^*, γ_2^* are γ_1, γ_2 given by Eqs. (33) and (34) where g, u are replaced by g_* and u_*, respectively.

The eigenvalues of the corresponding stability matrix are

$$\lambda_1 = 0, \tag{63}$$

$$\lambda_{2,3} = \frac{1}{2}[C \pm \sqrt{C^2 - 4D}], \tag{64}$$

where

$$C = 3\,\varepsilon + \chi_* \partial_\chi(\gamma_1 - \gamma_2)|_* + u_* \partial_u \gamma_1|_*,$$

$$D = 3\,\varepsilon \chi_* \partial_\chi(\gamma_1 - \gamma_2)|_* - \chi_* u_* (\partial_u \gamma_1 \partial_\chi \gamma_2 - \partial_\chi \gamma_1 \partial_u \gamma_2)|_*,$$

where $|_*$ means that the quantity must be taken at the fixed point. It can be shown numerically that for any positive values of u_* and for all possible values of the anisotropy parameters $\alpha_{1,2}$ the eigenvalues λ_2 and λ_3 are always greater than zero. Therefore, the corresponding fixed point is IR stable and satisfies the stability condition. It corresponds to the line $\varepsilon=\eta$ in Fig. 3, where the regions of stability for all possible fixed points are shown.

A detailed analysis of the scaling regimes and their physical meaning was done in Ref. [30], therefore we shall not repeat it here. We restrict our discussion only on the following basic conclusions: the trivial scaling regimes which are related to the fixed points FPI and FPIII have diffusive character and the others correspond to convective-type behavior. It is also important to stress that the Kolmogorov point ($\varepsilon=\eta=4/3$) lies on a boundary between two nontrivial regimes.

As was already mentioned (see the previous section) the issues of interest are especially multiplicatively renormalizable equal-time two-point quantities $G(r)$ (see also, e.g., Ref. [30]). Examples of such quantities are the equal-time structure functions in the inertial interval as they were defined in Eq. (21). The IR scaling behavior of the function $G(r)$ (for $r/l \gg 1$ and any fixed r/L),

$$G(r) \simeq \nu_0^{d_G^\omega} l^{-d_G} (r/l)^{-\Delta_G} R(r/L) \tag{65}$$

is related to the existence of IR stable fixed points of the RG equations (see above). In Eq. (65) d_G^ω and d_G are corresponding canonical dimensions of the function G (the canonical dimensions of the model are given in Sec. III), $R(r/L)$ is the so-called scaling function, which cannot be determined by the RG equation (see, e.g., Ref. [21]), and Δ_G is the critical dimension defined as

$$\Delta_G = d_G^k + \Delta_\omega d_G^\omega + \gamma_G^*. \tag{66}$$

Here γ_G^* is the fixed point value of the anomalous dimension $\gamma_G \equiv \mu \partial_\mu \ln Z_G$, where Z_G is the renormalization constant of the multiplicatively renormalizable quantity G, i.e., $G = Z_G G^R$ [31], and $\Delta_\omega = 2 - \gamma_\nu^*$ is the critical dimension of the frequency with $\gamma_\nu^* = \gamma_1^*$, which is defined in Eq. (33) and γ_1^* means that γ_1 is taken at the corresponding fixed point. From the above discussion of the possible scaling regimes we have

$$\gamma_\nu^* \equiv \xi = \begin{Bmatrix} 2\varepsilon - \eta & \text{for FPII} \\ \varepsilon & \text{for FPIV} \\ \varepsilon = \eta & \text{for FPV} \end{Bmatrix}. \tag{67}$$

We are working only in one-loop approximation but the anomalous dimension γ_ν^* is already exact for all fixed points at the one-loop level [30,34], i.e., it has no loop corrections of higher order, therefore the critical dimensions of frequency ω and of fields $\Phi \equiv \{\mathbf{v}, \theta, \theta'\}$ are also found exactly at one-loop level approximation [30]. In our notation they read

$$\Delta_\omega = 2 - \gamma_\nu^* = \begin{Bmatrix} 2 - 2\varepsilon + \eta & \text{for FPII} \\ 2 - \varepsilon & \text{for FPIV} \\ 2 - \varepsilon = 2 - \eta & \text{for FPV} \end{Bmatrix}, \tag{68}$$

and

$$\Delta_{\mathbf{v}} = 1 - \gamma_\nu^*, \quad \Delta_\theta = -1 + \gamma_\nu^*/2, \quad \Delta_{\theta'} = d + 1 - \gamma_\nu^*/2. \tag{69}$$

The renormalized function G^R must satisfy the RG equation of the form

$$(\mathcal{D}_{RG} + \gamma_G) G^R(r) = 0, \tag{70}$$

with operator $\mathcal{D}_{RG}$ given explicitly in Eq. (22), namely,

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \sum_{i=g,\chi,u} \beta_i \partial_i - \gamma_\nu \mathcal{D}_\nu. \tag{71}$$

The difference between the functions G and G^R is only in the normalization, choice of parameters (bare or renormalized), and related to this choice the form of the perturbation theory (in g_0 or in g). The existence of a nontrivial IR stable fixed point means that in the IR asymptotic region $r/l \gg 1$ and any fixed r/L the function $G(r)$ takes on the self-similar form given in Eq. (65). As was already mentioned the scaling function $R(r/L)$ is not determined by the RG equation itself. The dependence of the scaling functions on the argument r/L in the region $r/L \ll 1$ can be studied using the well-known Wilson operator product expansion (OPE) [20,21,23,24]. It shows that, in the limit $r/L \to 0$, the function $R(r/L)$ can be written in the following asymptotic form:

$$R(r/L) = \sum_i C_{F_i}(r/L)(r/L)^{\Delta_{F_i}}, \tag{72}$$

where C_{F_i} are coefficients regular in r/L. In general, the summation is implied over certain renormalized composite operators F_i with critical dimensions Δ_{F_i}. In the case under consideration the leading contribution is given by operators F_i having the form $F[N,p] = \partial_{i_1}\theta \cdots \partial_{i_p}\theta(\partial_i\theta\partial_i\theta)^n$ with $N = p + 2n$. In the next section we shall consider them in detail, where the complete one-loop calculation of the critical dimensions of the composite operators $F[N,p]$ will be presented for arbitrary values of N, d, u, and $\alpha_{1,2}$.

V. CRITICAL DIMENSIONS OF COMPOSITE OPERATORS AND ANOMALOUS SCALING

A. Operator product expansion

According to the OPE [20,21,23,24], the equal-time product $F_1(x')F_2(x'')$ of two renormalized composite operators [61] at $\mathbf{x} = (\mathbf{x}' + \mathbf{x}'')/2 = \text{const}$ and $\mathbf{r} = \mathbf{x}' - \mathbf{x}'' \to 0$ can be written in the following form:

$$F_1(x')F_2(x'') = \sum_i C_{F_i}(\mathbf{r}) F_i(\mathbf{x}, t), \tag{73}$$

where the summation is taken over all possible renormalized local composite operators F_i allowed by symmetry with definite critical dimensions Δ_{F_i}, and the functions C_{F_i} are the corresponding Wilson coefficients regular in L^{-2}. The renormalized correlation function $\langle F_1(x')F_2(x'')\rangle$ can now be found by averaging Eq. (73) with the weight $\exp S^R$ with S^R from Eq. (19). The quantities $\langle F_i \rangle$ appear on the right-hand side and their asymptotic behavior in the limit $L^{-1} \to 0$ is then found from the corresponding RG equations and has the form $\langle F_i \rangle \propto L^{-\Delta_{F_i}}$.

From the OPE (73) one can find that the scaling function $R(r/L)$ in the representation (65) for the correlation function $F_1(x')F_2(x'')$ has the form given in Eq. (72), where the coefficients C_{F_i} are regular in $(r/L)^2$.

It is well known that the specific feature of the turbulence models is the existence of operators with negative critical dimensions (the so-called "dangerous" operators) [21,23–25,28]. Their presence in the OPE determines the IR behavior of the scaling functions and leads to their singular dependence on L when $r/L \to 0$. At this point the turbulence models are crucially different from the models of critical phenomena, where the leading contribution to the representation (65) is given by the simplest operator $F = 1$ with the dimension $\Delta_F = 0$, and the other operators determine only the corrections that vanish for $r/L \to 0$. If the spectrum of the dimensions Δ_{F_i} for a given scaling function is bounded from below, the leading term of its behavior for $r/L \to 0$ is given by the minimal dimension. As was discussed in Ref. [30], the model under consideration belongs to this case for small enough values of the exponents ε, η.

In what follows, we shall concentrate on the equal-time structure functions of the scalar field as defined in Eq. (21).

The representation (65) is valid with the dimensions

$$d_G^{\omega} = -N/2, \quad d_G = -N, \quad \Delta_G = N\Delta_{\theta} = N(-1 + \gamma_{\nu}^{*}/2). \tag{74}$$

In general, not only do the operators which are present in the corresponding Taylor expansion enter into the OPE but also all possible operators that admix to them in renormalization. It can be shown by corresponding dimensional analysis that in the isotropic case the principal role is played by the composite operators $F_N = F_{2n} = (\partial_i\theta\partial_i\theta)^n$ (see, e.g., Ref. [30] for details). On the other hand, in the model with large- or small-scale anisotropy the leading contribution of the Taylor expansion for the structure functions (21) is given by the tensor composite operators constructed solely of the scalar gradients of the following form:

$$F[N,p] \equiv \partial_{i_1}\theta \cdots \partial_{i_p}\theta(\partial_i\theta\partial_i\theta)^n, \tag{75}$$

where $N=p+2n$ is the total number of the fields θ entering into the operator and p is the number of the free vector indices (see, e.g., Ref. [27]).

Composite operators $F[N,p]$: Renormalization and critical dimensions

Let us briefly discuss renormalization of the composite operators (75). A complete and detailed discussion of the renormalization of the composite operators is given in Ref. [26]. Therefore, we shall discuss only basic moments necessary to present explicit expressions for composite operators.

The necessity of additional renormalization of the composite operators (75) is related to the fact that the coincidence of the field arguments in Green functions containing them leads to additional UV divergences. These divergences must be removed by a special kind of renormalization procedure which can be found, e.g., in Refs. [19–21], where their renormalization is studied in general. As for the renormalization of composite operators in the models of turbulence, it is discussed in Refs. [23,24]. Besides, typically, the composite operators are mixed under renormalization. Therefore, let us briefly discuss this issue [21].

Let $F \equiv \{F_{\alpha}\}$ be a closed set of composite operators which are mixed only with each other in renormalization. Then the renormalization matrix $Z_F \equiv \{Z_{\alpha\beta}\}$ and the matrix of corresponding anomalous dimensions $\gamma_F \equiv \{\gamma_{\alpha\beta}\}$ for this set are given as follows:

$$F_{\alpha} = \sum_{\beta} Z_{\alpha\beta}F_{\beta}^{R}, \quad \gamma_F = Z_F^{-1}\tilde{D}_{\mu}Z_F. \tag{76}$$

Renormalized composite operators are subject to the following RG differential equations:

$$\left(\mathcal{D}_{\mu} + \sum_{i=g,\chi,u} \beta_i\partial_i - \gamma_{\nu}\mathcal{D}_{\nu}\right)F_{\alpha}^{R} = -\sum_{\beta}\gamma_{\alpha\beta}F_{\beta}^{R}, \tag{77}$$

which lead to the following matrix of critical dimensions $\Delta_F \equiv \{\Delta_{\alpha\beta}\}$:

$$\Delta_F = d_F^k + \Delta_{\omega}d_F^{\omega} + \gamma_F^{*}, \quad \Delta_{\omega} = 2 - \gamma_{\nu}^{*}, \tag{78}$$

where d_F^k and d_F^{ω} are diagonal matrices of corresponding canonical dimensions and γ_F^{*} is the matrix of anomalous dimensions (76) taken at the fixed point. In the end, the critical dimensions of the set of operators $F \equiv \{F_{\alpha}\}$ are given by the eigenvalues of the matrix Δ_F. The so-called "basis" operators that possess definite critical dimensions have the form

$$F_{\alpha}^{bas} = \sum_{\beta} U_{\alpha\beta}F_{\beta}^{R}, \tag{79}$$

where the matrix $U_F = \{U_{\alpha\beta}\}$ is such that $\Delta_F' = U_F\Delta_F U_F^{-1}$ is diagonal.

As was already mentioned, in what follows, the central role is played by the tensor composite operators $\partial_{i_1}\theta\cdots\partial_{i_p}\theta(\partial_i\theta\partial_i\theta)^n$. It is convenient to deal with the scalar operators obtained by contracting the tensors with the appropriate number of the uniaxial anisotropy vectors $\mathbf{n}$ [27],

$$F[N,p] \equiv [(\mathbf{n}\cdot\partial)\theta]^p(\partial_i\theta\partial_i\theta)^n, \quad N \equiv 2n + p. \tag{80}$$

A detailed analysis shows that the composite operators (80) with different N are not mixed in renormalization, and therefore the corresponding renormalization matrix $Z_{[N,p][N',p']}$ is in fact block diagonal, i.e., $Z_{[N,p][N',p']}=0$ for $N' \neq N$ [27].

In the isotropic case, as well as in the case when large-scale anisotropy is present, the elements $Z_{[N,p][N,p']}$ vanish for $p<p'$, thus the block $Z_{[N,p][N,p']}$ is in fact triangular along with the corresponding blocks of the matrices U_F and Δ_F from Eqs. (79) and (78). In the isotropic case it can be diagonalized by changing to irreducible operators (scalars, vectors, and traceless tensors), but even for nonzero imposed gradient its eigenvalues are the same as in the isotropic case. Therefore, the inclusion of large-scale anisotropy does not affect critical dimensions of the operators (80). On the other hand, in the case of small-scale anisotropy, the operators with different values of p mix heavily in renormalization, and the matrix $Z_{[N,p][N,p']}$ is neither diagonal nor triangular here and one can write

$$F[N,p] = \sum_{l=0}^{\lfloor N/2\rfloor} Z_{[N,p][N,N-2l]}F^{R}[N,N-2l], \tag{81}$$

where $\lfloor N/2\rfloor$ means the integer part of the $N/2$. Therefore, each block of renormalization constants with given N is a $(\lfloor N/2\rfloor+1)\times(\lfloor N/2\rfloor+1)$ matrix. Of course, the matrix of critical dimensions (78), whose eigenvalues at IR stable fixed point are the critical dimensions $\Delta[N,p]$ of the set of operators $F[N,p]$, has also dimension $(\lfloor N/2\rfloor+1)\times(\lfloor N/2\rfloor+1)$.

Now let us turn to the calculation of the renormalization constants $Z_{[N,p][N,p']}$ in the one-loop approximation in our model. We shall proceed as in Refs. [27,30]. Let $\Gamma(x;\theta)$ be the generating functional of the one-irreducible Green functions with one composite operator $F[N,p]$ from Eq. (80) and any number of fields θ. We shall be interested in the Nth term of the expansion of $\Gamma(x;\theta)$ in θ, which we denote $\Gamma_N(x;\theta)$; it has the form

$$\Gamma_N(x;\theta) = \frac{1}{N!}\int dx_1 \cdots \int dx_N\theta(x_1)\cdots\theta(x_N) \times \langle F[N,p] \times (x)\theta(x_1)\cdots\theta(x_N)\rangle_{one\text{-}ir}, \tag{82}$$

and in the one-loop approximation it is given as

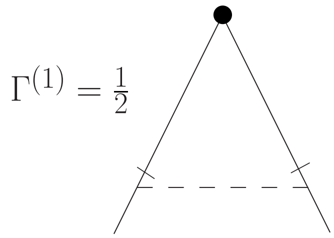

FIG. 4. Graphical representation of the one-loop correction to Γ_N in Eq. (83).

$$\Gamma_N = F[N,p] + \Gamma^{(1)}, \tag{83}$$

where $\Gamma^{(1)}$ is given by the analytical calculation of the diagram in Fig. 4, and the first term in Eq. (83) represents "tree" approximation (see also Ref. [27]).

The black circle with two attached lines in the diagram in Fig. 4 denotes the variational derivative $V(x;x_1,x_2) \equiv \delta^2 F[N,p]/\delta\theta(x_1)\delta\theta(x_2)$, where the second variation makes needed combinatorics, namely, the operator $F[N,p]$ contains N fields θ and one must take two of them (in all possible ways) to construct the one-loop diagram as it is shown in Fig. 4. It can be represented in the following convenient form [27]:

$$V(x;x_1,x_2) = \partial_i\delta(x-x_1)\partial_j\delta(x-x_2) \times \frac{\partial^2}{\partial a_i\,\partial a_j}[(\mathbf{na})^p(a^2)^n], \tag{84}$$

where a constant vector a_i will be substituted with $\partial_i\theta(x)$ after the differentiation. The analytical form of the diagram in Fig. 4 (without the symmetry factor $1/2$) is the following:

$$\int dx_1 \cdots \int dx_4 V(x;x_1,x_2)\langle\theta(x_1)\theta'(x_3)\rangle_0 \times \langle\theta(x_2)\theta'(x_4)\rangle_0\langle v_k(x_3)v_l(x_4)\rangle_0\partial_k\theta(x_3)\partial_l\theta(x_4), \tag{85}$$

where the bare propagators are given in Eqs. (14), (15) and the derivatives are related to the ordinary vertex factors shown in Fig. 1.

We are interested in the UV divergent part of the expression (85), which is needed for determination of the corresponding renormalization constants. But the needed UV divergent part is proportional to the polynomial built of N gradients $\partial_i\theta(x)$ at a single space-time point x, and all of them have been already extracted from Eq. (85), namely, $N-2$ gradients are given by the vertex (84) and the other two gradients are given by the ordinary vertex factors in Fig. 1. This important point from the view of calculations allows us to replace the gradients with the constant vectors $\mathbf{a}$. Therefore, in the end, the divergent part of expression (85) can be written in the following compact form:

$$a_k a_l \frac{\partial^2}{\partial a_i\,\partial a_j}[(\mathbf{na})^p(a^2)^n]X_{ij,kl}, \tag{86}$$

with

$$X_{ij,kl} \equiv \int dx_3 \int dx_4 \partial_i\langle\theta(x)\theta'(x_3)\rangle_0 \times \partial_j\langle\theta(x)\theta'(x_4)\rangle_0\langle v_k(x_3)v_l(x_4)\rangle_0, \tag{87}$$

or, in the momentum-frequency representation (suitable for the further calculations), after integration over the frequency,

$$X_{ij,kl} = \frac{D_0}{2u_0^2\nu_0^3}\int \frac{d\mathbf{k}}{(2\pi)^d}\frac{k_ik_j}{(k^2+m^2)^{d/2+\varepsilon}}T_{kl}(\mathbf{k}) \times \left(\frac{1}{k^2+\chi(\mathbf{nk})^2} - \frac{1}{k^2(1+u)+\chi(\mathbf{nk})^2}\right), \tag{88}$$

with D_0 from Eq. (4) and T_{kl} from Eq. (10) (we again use the possibility to work with $\eta=0$ within one-loop approximation [30,31]). Expression (88) can be decomposed into some tensor structures (see, e.g., Ref. [27]) and after rather long but direct calculations we are coming to the following result for the quantity defined in Eq. (86):

$$\frac{S_d}{(2\pi)^d}\frac{g}{4u^2}\left(\frac{\mu}{m}\right)^{2\varepsilon}\frac{1}{\varepsilon}\{Q_1F[N,p-2] + Q_2F[N,p] + Q_3F[N,p+2] + Q_4F[N,p+4]\}, \tag{89}$$

where we have substituted the unrenormalized quantities with the renormalized one, a_i have been replaced with the gradients $\partial_i\theta(x)$ (thus they again form the operators $F[N,q]$, with $q=p-2,p,p+2,p+4$), and the following notation was applied for the corresponding coefficients:

$$Q_i = \sum_{j=0}^{3} A_{ij}\left(H_j - \frac{1}{1+u}G_j\right), \quad i=1,\ldots,4, \tag{90}$$

where H_j and G_j are the hypergeometric functions of the following form:

$$H_j = {}_2F_1\left(\frac{1}{2},1;j+\frac{d}{2};-\chi\right),$$

$$G_j = {}_2F_1\left(\frac{1}{2},1;j+\frac{d}{2};-\frac{\chi}{1+u}\right),$$

with $j=0,\ldots,3$, and coefficients A_{ij} for $i=1,\ldots,4$ and $j=0,\ldots,3$ are given in the Appendix.

Using the standard renormalization procedure the renormalization constants $Z_{[N,p][N,p']}$ defined in Eq. (81) are found from the requirement that function (83) is UV finite (contains no poles in ε) when it is written in renormalized variables and with the replacement $F[N,p] \to F^R[N,p]$. In the end, from Eqs. (83) and (89) we have

$$Z_{[N,p][N,p-2]} = \frac{\bar{g}}{8u^2\varepsilon}Q_1, \tag{91}$$

$$Z_{[N,p][N,p]} = 1 + \frac{\bar{g}}{8u^2\varepsilon}Q_2, \tag{92}$$

$$Z_{[N,p][N,p+2]} = \frac{\bar{g}}{8u^2\varepsilon} Q_3, \tag{93}$$

$$Z_{[N,p][N,p+4]} = \frac{\bar{g}}{8u^2\varepsilon} Q_4, \tag{94}$$

with coefficients Q_i given in Eq. (90). Using the definition of the matrix of anomalous dimensions $\gamma_{[N,p][N',p']}$ given in Eq. (76) we are coming to the following result:

$$\gamma_{[N,p][N,p-2]} = -\frac{\bar{g}}{4u^2} Q_1,$$

$$\gamma_{[N,p][N,p]} = -\frac{\bar{g}}{4u^2} Q_2,$$

$$\gamma_{[N,p][N,p+2]} = -\frac{\bar{g}}{4u^2} Q_3,$$

$$\gamma_{[N,p][N,p+4]} = -\frac{\bar{g}}{4u^2} Q_4, \tag{95}$$

and the desired matrix of critical dimensions (78) has the form

$$\Delta_{[N,p][N,p']} = N\gamma_\nu^*/2 + \gamma^*_{[N,p][N,p']}, \tag{96}$$

where the asterisk means that the quantities are taken at the corresponding fixed point (see Sec. IV) and γ_ν^* is given in Eq. (67). The nonzero one-loop contribution to the matrix of critical dimension (96) is represented by Eqs. (95) with Q_i, $i=1,\ldots,4$ defined in Eq. (90). It means that the matrix elements of the matrix $\gamma_{[N,p][N',p']}$ other than given in Eq. (95) are equal to zero. It can be seen immediately that the matrix of critical dimensions depends on the anisotropy parameters α_1 and α_2 and, what is now more interesting and important here, on the parameter u (see below).

In the end, the critical dimensions $\Delta[N,p]$ are given by the eigenvalues of the matrix (96). The simplest situation occurs in the isotropic limit with $\alpha_1=\alpha_2=0$ and, correspondingly, $\chi^*=0$. In this case, one comes to the triangular matrix, therefore its eigenvalues are given directly by the diagonal elements. But more interesting is the fact that within the isotropic model we have the same eigenvalues of the matrix of critical dimensions for all fixed point values of u^*, i.e., the eigenvalues are independent of u at the fixed point, namely,

$$\Delta[N,p] = \frac{N\xi}{2} + \frac{2p(p-1)-(d-1)(N-p)(d+N+p)}{2(d-1)(d+2)}\xi, \tag{97}$$

where ξ is given in Eq. (67) (see, e.g., Ref. [30] for details). As a result, it means that within the one-loop approximation there is no difference between the general model with finite time correlations and its two special limits, namely, Kraichnan's rapid-change limit and the frozen limit of the model as for the anomalous behavior of the equal-time structure functions (it, of course, also holds for the other equal-time correlation functions).

The situation is different when the presence of small-scale anisotropy is supposed. In this case, the matrix of critical dimensions is not diagonal and the eigenvalues depend on anisotropy parameters, as well as on the parameter u. It leads to the sufficient difference between anomalous dimensions of the models with different time correlations of the velocity field. On the other hand, the fact that the matrix (96) is triangular in the isotropic case (it is also triangular in the case with large-scale anisotropy) is also important here because it allows us to assign uniquely the concrete critical dimension to the corresponding composite operator even in the case with small-scale anisotropy and study their hierarchical structure as functions of p (see Ref. [27] for details). As was shown in Ref. [27] within the Kraichnan model, as for anomalous scaling, the leading role is played by the operators with the most negative critical dimensions: for the structure functions (21) with even N it is the operator with $p=0$ and for the structure functions (21) with odd N it is the operator with $p=1$. As we shall see below, the same situation also holds in the general case with the finite time correlations.

Let us discuss the behavior of the critical dimension in the presence of small-scale anisotropy in detail. Our aim is twofold. First of all, we shall find the dependence of the critical dimensions on the parameter u, thus we shall answer the question whether the system with the finite time correlations of the velocity field with the presence of small-scale anisotropy is more anomalous, i.e., whether the corresponding critical dimensions are less than those of the Kraichnan rapid-change model, which was investigated in Ref. [27]. This question is interesting because the model with the finite correlation time of velocity field can be considered as a further step on the way to the model with a velocity field driven by the stochastic Navier-Stokes equation. Thus, the answer on the aforementioned question in the framework of the present model can also give a preliminary answer, as for possible tendencies, on the similar question in the framework of the scalar advection by the Navier-Stokes velocity field. The second aim is to investigate whether the system with the finite correlation time of velocity field together with the presence of small-scale anisotropy can lead to the more complicated structure of critical dimensions than it was shown in Ref. [27]. There are two possibilities. First, it is possible that the pairs of complex conjugate eigenvalues of the matrix of critical dimensions can exist. In this case, the oscillation behavior of the corresponding scaling function appears. Therefore, the scaling functions in Eq. (104) would contain terms of the following form:

$$(r/L)^{\Delta_R}\{c_1 \cos[\Delta_I(r/L)] + c_2 \sin[\Delta_I(r/L)]\}, \tag{98}$$

where Δ_R and Δ_I are the real and imaginary parts of Δ, and $c_{1,2}$ are constants. Another, in general, possible structure of the matrix (96) is related to the situation when the matrix of critical dimensions cannot be diagonalized and has only the Jordan form. Then a logarithmic correction would be involved to the powerlike behavior of the form

$$(r/L)^{\Delta}[c_1 \ln(r/L) + c_2], \tag{99}$$

where Δ is the eigenvalue related to the Jordan cell.

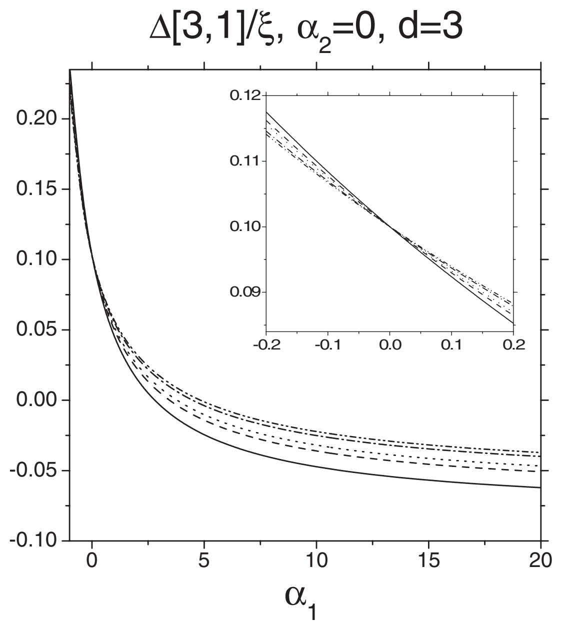

FIG. 5. Dependence of the critical dimension $\Delta[3,1]/\xi$ on anisotropy parameter α_1 ($\alpha_2=0$) for different fixed point values of the parameter u: $u^*=0$ (frozen limit)—solid line, $u^*=0.5$—dashed line, $u^*=1$—dotted line, $u^*=5$—dashed-dotted line, $u^*=\infty$ (rapid-change model limit)—dashed dotted dot line. The small figure shows details that are not visible in the basic figure.

In Figs. 5–14, the behavior of the minimal eigenvalues of the matrix of critical dimensions $\Delta[N,p]$ for various values of $N=3,4,5,6,7$ ($p=0$ for even values of N and $p=1$ for odd values of N) is shown as a function of the anisotropy parameters α_1 and α_2 in a three-dimensional case and for different fixed point values of the parameter u. The dependence of the critical dimension $\Delta[2,0]$ is not shown explicitly because it is identically equal to zero for all fixed point values of the parameter u. It can be shown either by direct calculation or by using the Schwinger equation (see, e.g., Ref. [27]). At first sight one can conclude that there are different behaviors of critical dimensions as functions of anisotropy parameters α_1 and α_2 and of the parameter u^* for odd and even structure functions. Let us discuss it in detail.

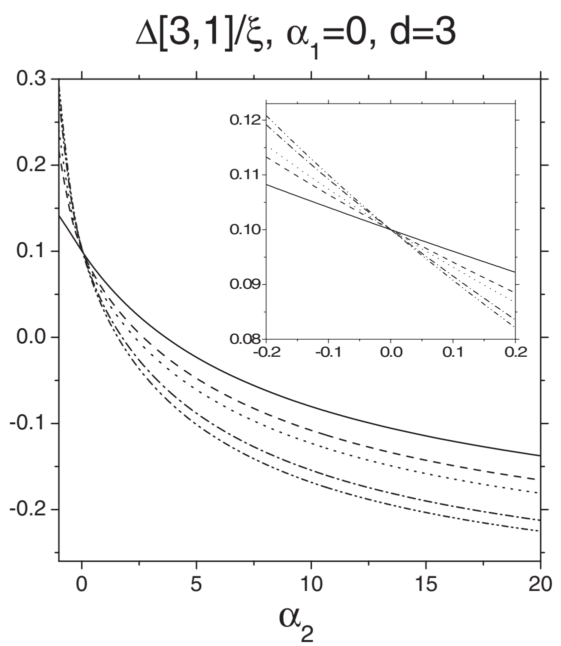

FIG. 6. Dependence of the critical dimension $\Delta[3,1]/\xi$ on anisotropy parameter α_2 ($\alpha_1=0$) for different fixed point values of the parameter u (for notation see the caption in Fig. 5).

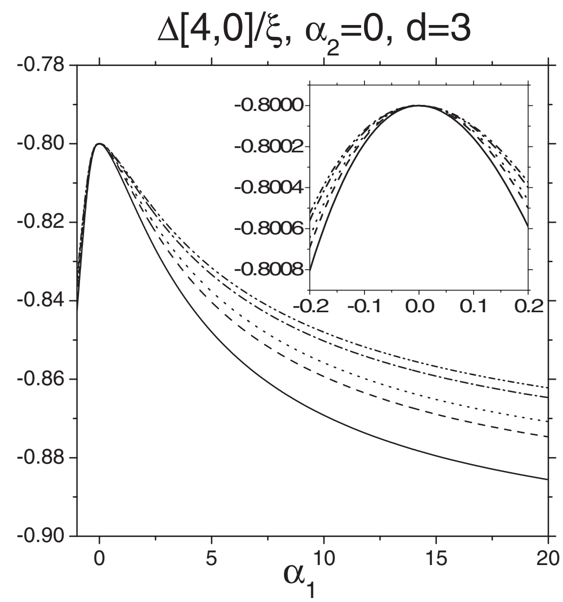

FIG. 7. Dependence of the critical dimension $\Delta[4,0]/\xi$ on anisotropy parameter α_1 ($\alpha_2=0$) for different fixed point values of the parameter u (for notation see the caption in Fig. 5).

For the composite operators $F[N,p]$ with even N ($N=2,4,6$) the minimal critical dimensions are related with the isotropic shell, i.e., with $p=0$. As was already mentioned, in

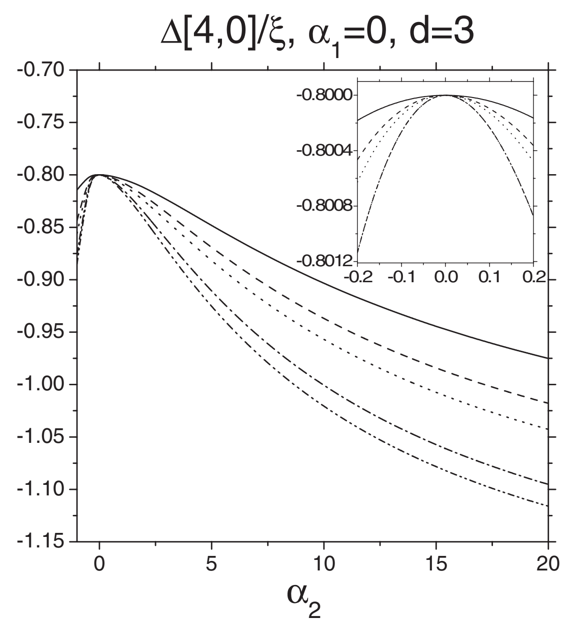

FIG. 8. Dependence of the critical dimension $\Delta[4,0]/\xi$ on anisotropy parameter α_2 ($\alpha_2=0$) for different fixed point values of the parameter u (for notation see the caption in Fig. 5).

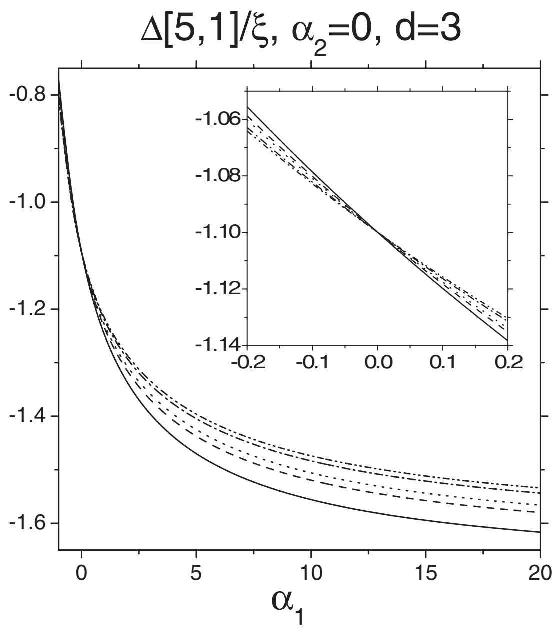

FIG. 9. Dependence of the critical dimension $\Delta[5,1]/\xi$ on anisotropy parameter α_1 ($\alpha_2=0$) for different fixed point values of the parameter u (for notation see the caption in Fig. 5).

the case $N=2$ the corresponding critical dimensions $\Delta[2,0]$ are identically equal to zero. On the other hand, one can see identical qualitative behavior of the critical dimensions $\Delta[4,0]$ and $\Delta[6,0]$ as functions of anisotropy parameters as it is shown in Figs. 7 and 8 and in Figs. 11 and 12. In the case when the anisotropy parameter α_2 is vanished the corresponding critical dimensions (as functions of parameter α_1) are the most negative in the frozen limit of the model ($u^*=0$) as is shown in Figs. 7 and 11. On the other hand, in the case when the anisotropy parameter α_1 is vanished (see Figs. 8 and 12) the situation is opposite, namely, the most negative critical dimensions as functions of the corresponding anisotropy parameters are those that correspond to the rapid-change model limit ($u^* \to \infty$). It is some kind of nonuniversality of the behavior of the critical dimensions in the plane of anisotropy parameters α_1-α_2. Thus, we still have the hierarchical behavior with respect to u^* but the hierarchy depends also on the values of anisotropy parameters. It means

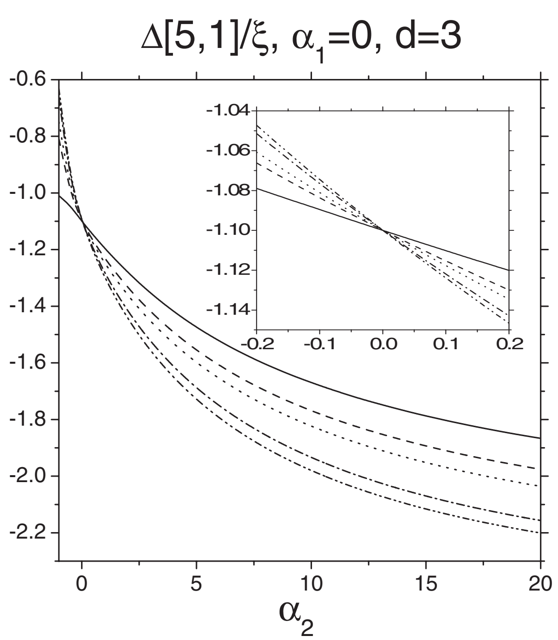

FIG. 10. Dependence of the critical dimension $\Delta[5,1]/\xi$ on anisotropy parameter α_2 ($\alpha_1=0$) for different fixed point values of the parameter u (for notation see the caption in Fig. 5).

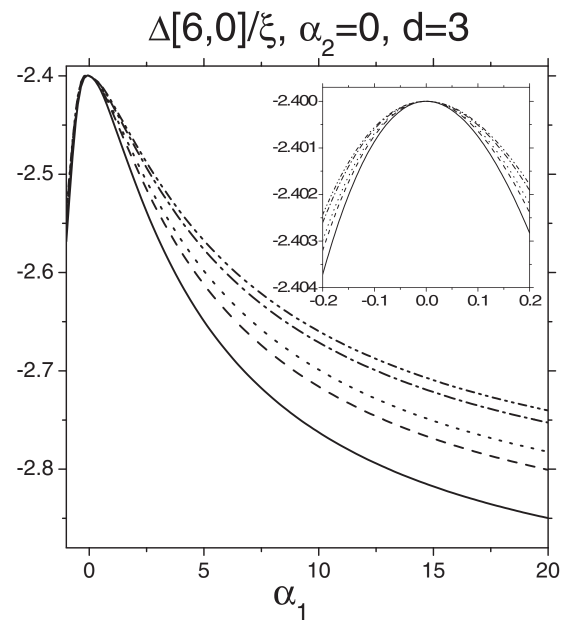

FIG. 11. Dependence of the critical dimension $\Delta[6,0]/\xi$ on anisotropy parameter α_1 ($\alpha_2=0$) for different fixed point values of the parameter u (for notation see the caption in Fig. 5).

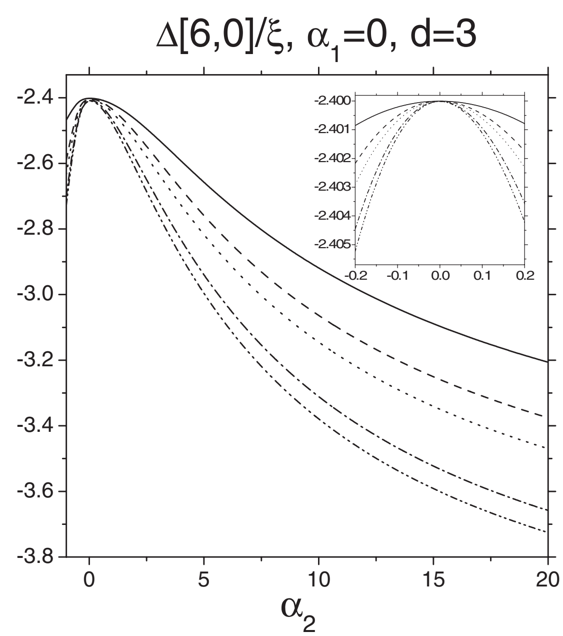

FIG. 12. Dependence of the critical dimension $\Delta[6,0]/\xi$ on anisotropy parameter α_2 ($\alpha_1=0$) for different fixed point values of the parameter u (for notation see the caption in Fig. 5).

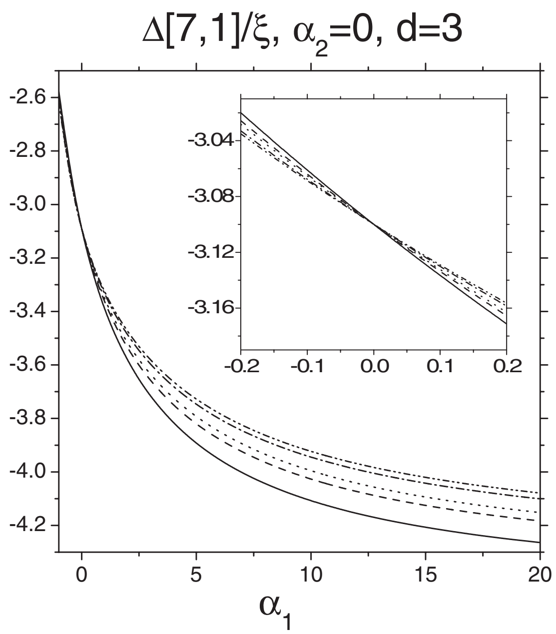

FIG. 13. Dependence of the critical dimension $\Delta[7,1]/\xi$ on anisotropy parameter α_1 ($\alpha_2=0$) for different fixed point values of the parameter u (for notation see the caption in Fig. 5).

physically that the answer to the question, which model is "more anomalous" can depend on the form of the small-scale anisotropy. Besides, it is evident that there must exist a system of curves in the plane α_1-α_2 on which the pairs of models with different fixed point values of the parameter u have the same anomalous dimensions. We shall not show them explicitly here because we suppose that their form will strongly depend on the higher-loop calculations which are ignored here (we work in one-loop approximation) but we can assume that the qualitative picture will be the same. Of course, all of the curves must cross in the point $\alpha_1=\alpha_2=0$ (as is evident from corresponding figures for the same value of N) as a result of the fact that in the isotropic case the critical dimensions for different values of u^* are the same and they are given explicitly in Eq. (97).

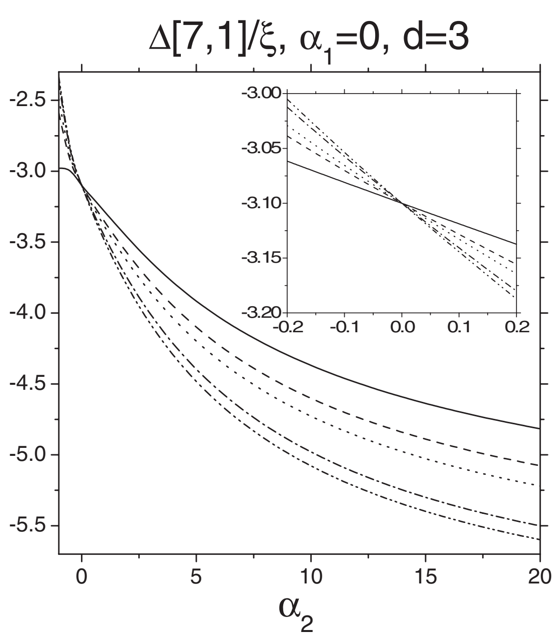

FIG. 14. Dependence of the critical dimension $\Delta[7,1]/\xi$ on anisotropy parameter α_2 ($\alpha_1=0$) for different fixed point values of the parameter u (for notation see the caption in Fig. 5).

As for the composite operators of odd order the situation is slightly different. Now, the minimal critical dimensions are related to $p=1$. Their behavior is shown in Figs. 5, 6, 9, 10, 13, and 14 for $N=3$, $N=5$, and $N=7$, respectively. One can see immediately that again in different directions in the α_1-α_2 plane different models are the most anomalous (frozen limit of the model or rapid-change model limit), i.e., they have the most negative critical dimensions $\Delta[N,1]$. But, besides, the situation is also different for positive and negative values of the anisotropy parameters. For example, in the case when the anisotropy parameter $\alpha_2=0$ the corresponding critical dimensions as functions of parameter α_1 are the most negative in the frozen limit of the model ($u^*=0$) for $\alpha_1>0$ and they are the most negative in the rapid-change model limit of the model ($u^*=\infty$) for $-1<\alpha_1<0$ as is shown in Figs. 5, 9, and 13. On the other hand, in the case when the anisotropy parameter $\alpha_1=0$ (see Figs. 6, 10, and 14) the situation is opposite. Thus, one can conclude that the answer to the question, whichmodel is more anomalous, can depend on the form of the small-scale anisotropy, i.e, on the parameters of anisotropy.

Further, the dimensions $\Delta[N,p]$ obey the following important hierarchies:

$$\Delta[2n,0] > \Delta[2n+2,0], \tag{100}$$

$$\Delta[2n+1,1] > \Delta[2n+3,1], \tag{101}$$

$$\Delta[N,p] > \Delta[N,p'], \quad p > p'. \tag{102}$$

In the isotropic case ($\alpha_{1,2}=0$), their validity follows from Eq. (97). On the other hand, in the small-scale anisotropic case they are results of numerical investigations [see Figs. 5–14 to test relations (100) and (101)]. Relations (100)–(102) will be important for the determination of asymptotic behavior of single-time structure functions in the next subsection.

In the present paper we show only the smallest critical dimension for a concrete value of N, namely, $p=0$ for even value of N and $p=1$ for odd value of N. But corresponding analysis can be also done for others critical dimensions which correspond to higher possible values of p ($p\leq N$). A detailed analysis shows that no exotic situation appears in their behavior as well. Thus, we can also answer the second question whether the finite correlation time of velocity field together with small-scale anisotropy can lead to the more complicated structure of critical dimensions (oscillations or logarithmic corrections). Our answer is no, i.e, the matrices of critical dimensions have real eigenvalues at least up to $N=7$.

B. Anomalous scaling of the structure functions in one-loop approximation

Now we have all the necessary results to write the final asymptotic expression for the structure functions (21). First of all, in the uniaxial anisotropy situation when the preferred direction is given by unit constant vector $\mathbf{n}$, as defined in Eq. (10), the structure functions (21) can be decomposed in the following way (Legendre decomposition) [36]:

$$S_N(\mathbf{r}) = \sum_{p=0}^{\infty} S_{N,p}(r)P_p(z), \quad z = (\mathbf{n}\cdot\mathbf{r})/r, \tag{103}$$

where $P_p(z)$ are the Gegenbauer polynomials, and $S_{N,p}$ are corresponding scalar coefficients which depend only on $r = |\mathbf{r}|$ in the case of large-scale anisotropy [30] but they can depend also on the anisotropy parameters in the case with small-scale anisotropy. Now, using the combination of the RG representation (65) for decomposed S_N given in Eq. (103) with dimensions (74) together with the OPE (72) leads to the general asymptotic expression for the structure functions (21) within the inertial range, namely,

$$S_N(\mathbf{r}) \simeq r^{N(1-\xi/2)} \times \sum_{N'\le N} \sum_p \{C_{N',p}(r/L)^{\Delta[N',p]} + \cdots\}, \tag{104}$$

where ξ is defined in Eq. (67), p obtains all possible values for given N', $C_{N',p}$ are numerical coefficients which are functions of the parameters of the model $(\xi, d, u, \alpha_1, \alpha_2, z)$, the dimensions $\Delta[N',p]$ are given by the eigenvalues of the matrix of critical dimensions (96), and the Gegenbauer polynomials $P_p(z)$ from decomposition (103) are included in the coefficients $C_{N',p}$. In Eq. (104) dots mean contributions by the operators other than $F[N,p]$, which are not important in the asymptotic regimes (see, e.g., [21,27] for details). The appearance of dimensions $\Delta[N',p]$ with $p \neq 0$ on the right hand of Eq. (104) is related to the fact that, in the presence of small-scale anisotropy, the corresponding operators acquire nonzero mean value. On the other hand, the leading term for the small r/L behavior of the structure function S_N is given by a term with the minimal possible value of $\Delta[N',p]$. Using the relations given in Eqs. (100)–(102) the final asymptotic expression for the single-time structure functions in the presence of weak uniaxial small-scale anisotropy is

$$S_N(\mathbf{r}) \simeq r^{N(1-\xi/2)}(r/L)^{\Delta[N,0]}, \tag{105}$$

for an even value of N, and

$$S_N(\mathbf{r}) \simeq r^{N(1-\xi/2)}(r/L)^{\Delta[N,1]}, \tag{106}$$

for an odd value of N.

In the end, let us briefly discuss the inertial-range behavior of the skewness. In our case it has the following asymptotic form:

$$\frac{S_3}{S_2^{3/2}} \simeq \left(\frac{r}{L}\right)^{\Delta[3,1]}, \tag{107}$$

which, on one hand, coincides with the result of Refs. [17] in the isotropic limit ($\alpha_{1,2}=0$) and, on the other hand, coincides also with the anisotropic Kraichnan model limit ($u^* \to \infty$) that was obtained in Ref. [27].

VI. CONCLUSION

Using the field theoretic RG technique and operator product expansion we have investigated the influence of uniaxial small-scale anisotropy on a passive scalar advected by a Gaussian solenoidal velocity field with finite correlation time in one-loop approximation. First of all we have found and classified all possible scaling regimes of the model which are directly related to the corresponding IR stable fixed points of the RG equations. The "phase diagram" of the scaling regimes in the plane ε-η is shown (see Fig. 3) and it is found that the small-scale anisotropy has no influence on the stability of the scaling regimes (at a one-loop level), i.e., we have the same five scaling regimes with the same regions of stability as in the isotropic case of the model [30]. Two of the scaling regimes are related to the "frozen limit" of the model, another two to the "rapid-change" model, and the last one corresponds to the general case with finite time correlations of the velocity field.

Further, we have studied the influence of small-scale anisotropy on the anomalous scaling of the single-time structure functions of a passive scalar using the OPE. The corresponding leading composite operators with the smallest (the most negative) critical dimensions are studied in detail and the critical dimensions are found as functions of the anisotropy parameters and the fixed point value of the parameter u, which represents the ratio of turnover time of scalar field and velocity correlation time. We have shown that the corresponding anomalous dimensions, which are the same (universal) for all particular models with a concrete value of u in the isotropic case, are different (nonuniversal) in the case with the presence of small-scale anisotropy and they are continuous functions of the anisotropy parameters, as well as the parameter u. It is shown that there is different behavior of the anomalous dimensions in the case of even order single-time structure functions than in the case of odd order ones, as well as there is different behavior of the anomalous dimensions in the different directions in the plane of the anisotropy parameters (see the discussion in the end of the previous section for details). Thus, the answer to the question, which special case of the general model (rapid-change limit or frozen limit) is more anomalous in the presence of anisotropy, is not unique. Therefore, we are also not able to make a definite conclusion about what one can expect in the case of a more realistic model of a passive scalar advection, namely, in the model of a passive scalar advected by the Navier-Stokes velocity field.

It was also shown that even in the case with finite time correlations of the velocity field the critical dimensions of the corresponding composite operators have simple structure, i.e., the matrices of the critical dimensions have real eigenvalues. It means that no exotic situations, namely, oscillations or logarithmic corrections to the critical dimensions, are present.

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APPENDIX

The explicit form of the coefficients A_{ij}, with $i=1,\ldots,4$ and $j=0,\ldots,3$ from Eq. (90) is

$$A_{10}=\frac{p(p-1)[(d^2-5)(1+\alpha_1)+4\alpha_2]}{d^2-1},$$

$$A_{11}=-\frac{p(p-1)}{d(d+1)}[d^2+d-4-\alpha_2(d-7)+\alpha_1(2d^2+d-9)],$$

$$A_{12}=\frac{p(p-1)}{d(d+2)}[d+1+(d-1)(\alpha_1(d+3)-2\alpha_2)],$$

$$A_{13}=p(p-1)(\alpha_2-\alpha_1)\frac{(d+1)(d+3)}{d(d+2)(d+4)},$$

$$A_{20}=\frac{1}{d^2-1}\{48n(n-1)(\alpha_2-\alpha_1-1)-p(p-1)[16\alpha_2+(1+\alpha_1)(7+d^2)]+2n[4\alpha_2(d+2)+(1+\alpha_1)(d^2-4d+9)+2p(4\alpha_2+(1+\alpha_1)(d^2-13))]\},$$

$$A_{21}=\frac{1}{d(d+1)}\{4n(n-1)[d+13-24\alpha_2+(d+25)\alpha_1]+p(p-1)[2d^2+d+7+\alpha_2(7-d)+\alpha_1(3d^2+d+14)]-2n[2p(d^2+3d-10)-3d-7+\alpha_2(2p(15-d)+7d+15)+\alpha_1(2p(2d^2+3d-23)+d^2-7d-16)]\},$$

$$A_{22}=-\frac{1}{d(d+2)}\{2n(\alpha_1-\alpha_2)(7+3d)+n(n-1)[12-60\alpha_2+4\alpha_1(16+d)]+np[16\alpha_2(d-2)-4\alpha_1(d-1)(d+7)-12(d+1)]+p(p-1)[d(d+1)-\alpha_2(d^2+2d+9)+\alpha_1(3d^2+2d+7)]\},$$

$$A_{23}=\frac{(d+3)(\alpha_1-\alpha_2)}{d(d+2)(d+4)}\{12n(n-1)+p(d+1)[d(p-1)-12n]\},$$

$$A_{30}=\frac{2n}{d^2-1}\{2(n-1)[(d^2-1)(1+\alpha_1)-24\alpha_2]-(2p+1)(d^2-1)(1+\alpha_1)-8\alpha_2(d+2+4p)\},$$

$$A_{31}=\frac{2n}{d(d+1)}\{2(n-1)[(d+1)(d+6)-\alpha_2(d+25)+\alpha_1(d+1)(2d+5)]+(d+1)(d+2p(2d+1))+\alpha_2(15+7d+2p(15-d))+\alpha_1(d+1)(d(2+6p)-1)\},$$

$$A_{32}=\frac{2n}{d(d+2)}\{2(n-1)[\alpha_1(18+d(d+13))-6(\alpha_2-1)(d+2)]-(d+1)[d(\alpha_1-\alpha_2)+2(d+2+3\alpha_1(d+1)-\alpha_2(d+3))p]\},$$

$$A_{33}=\frac{4n(\alpha_1-\alpha_2)(d+3)[(d+1)p-6(n-1)]}{d(d+4)},$$

$$A_{40}=-4n(n-1)(1+\alpha_1),$$

$$A_{41}=\frac{4n(n-1)[2d+4-\alpha_2+3\alpha_1(d+1)]}{d},$$

$$A_{42}=-\frac{4n(n-1)[3\alpha_1(d+2)-(\alpha_2-1)(d+4)]}{d},$$

$$A_{43}=\frac{4n(n-1)(\alpha_1-\alpha_2)(d+3)}{d},$$

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[61] By definition we use the term "composite operator" for any local monomial or polynomial constructed from primary fields and their derivatives at a single point $x \equiv (t, \mathbf{x})$. Constructions $\theta^n(x)$ and $[\partial_i \theta(x) \partial_i \theta(x)]^n$ are typical examples.